Rational functions

Thursday, September 21, 2023 2:12 PM

Rational functions

$R(z) = \frac{P(z)}{Q(z)}$, P, Q - polynomials. Assume - no common zeroes (can factor them out).

$R'(z) = \frac{P'(z)\,Q(z) - Q'(z)\,P(z)}{Q(z)^2}$ - exists when z is not a pole.

Def. Zeroes of $Q(z)$ are called poles of $R(z)$.
Order (multiplicity) of a pole = order of zero of $Q(z)$.

Example. $R(z) = \frac{(z-i)^2(z+1)}{(z+i)^3}$ has pole $-i$ of order 3, zero -1 of order 1, zero i of order 2.

Remark. If z_0 - pole of $R(z)$, then $\lim_{z \to z_0} R(z) = \infty$ (in spherical metric)
So we put $R(z_0) = \infty$

Behavior at ∞: Consider $R_1(z) := R(\frac{1}{z}) = \frac{P(\frac{1}{z})}{Q(\frac{1}{z})}$.

$\frac{1}{z}$ - trick. $R(\infty) := R_1(0)$. $\lim_{z \to \infty} R(z) = \lim_{z \to 0} R_1(z)$

More details: if $P(z) = \sum_{k=0}^{n} a_k z^k$, $Q(z) = \sum_{k=0}^{m} b_k z^k$, $n = \deg P$, $m = \deg Q$.

then $R_1(z) = z^{m-n} \underbrace{\frac{\sum_{k=0}^{n} a_k z^{n-k}}{\sum_{k=0}^{m} b_k z^{m-k}}}_{R_0(z)}$. $a_n \neq 0$, $b_m \neq 0$.

$P(\frac{1}{z}) = \sum_{k=0} a_k z^{-k} = z^{-n} \sum_{k=0}^{n} a_k z^{n-k}$

$Q(\frac{1}{z}) = z^{-m} \sum_{k=0}^{m} b_k z^{m-k}$

$R_0(0) = \frac{a_n}{b_m} \neq 0, \infty$.

So $R(\infty) = R_1(0) = \begin{cases} \frac{a_m}{b_n}, & \deg P = \deg Q \\ 0, & \deg P < \deg Q \\ \infty, & \deg P > \deg Q. \end{cases}$

If $\deg P < \deg Q$, ∞ - zero of $R(z)$, order of 0 is the order of 0 as zero of $R_1(z)$, i.e. $\deg Q - \deg P$

If $\deg P > \deg Q$, ∞ is a pole of $R(z)$ of $\deg P - \deg Q$.

If $\deg P = \deg Q$, ∞ is neither pole nor zero.

$\deg P > \deg Q$: total number of zeroes, counting order =
\# zeroes of P + \# zeroes at ∞ = $\deg P$
total number of poles, ... =
\# zeroes of Q + \# poles at ∞ = $\deg Q + \deg P - \deg Q =$

deg P.

So, the total number of zeroes = the total number of poles = max(deg P, deg Q).

Def. max(deg P, deg Q) is called order of $R(z)$.

Remark. $\forall w \in \mathbb{C}$, the equation $R(z) = w$ has deg R roots, counting multiplicity, since $R(z) - w$ has the same poles as $R(z)$.

Order = 1: Möbius (Linear) maps: $\frac{az+b}{cz+d}$ $ad - bc \neq 0$, $a, b, c, d \in \mathbb{C}$

Polynomials are the rational functions with all the poles at ∞.

Partial fraction decomposition.

Singular part at ∞: If $R(z) = \frac{P(z)}{Q(z)}$ and deg P > deg Q, let $P(z) = G_\infty(z)Q(z) + S(z)$, deg S < deg Q,

so $R(z) = G_\infty(z) + \underbrace{\frac{S(z)}{Q(z)}}_{H(z)}$, $H(\infty) = 0$.

$G_\infty(z)$ - polynomial, $\deg G_\infty(z) > 0$, so $G_\infty(\infty) = \infty$.

$G_\infty(z)$ - singular part of R at ∞.

If deg P < deg Q, let $G_\infty(z) \equiv 0$.

If deg P = deg Q, let $G_\infty(z) \equiv R(\infty)$ - a constant.

Observe! $\deg G = \max(\deg P - \deg Q, 0)$ - order of ∞ as pole.

Let z_0 be a pole of R. Consider $R_1(\zeta) := R(z_0 + \frac{1}{\zeta})$ - a rational function, $R_1(\infty) = R(z_0) = \infty$, so

$R_1(\zeta) = \frac{P_1(\zeta)}{Q_1(\zeta)}$, $\deg P_1 > \deg Q_1$. Let $G_{z_0}(\zeta)$ - singular part of $R_1(\zeta)$ at ∞, so $R_1(\zeta) = G_{z_0}(\zeta) + H_{z_0}(\zeta)$, $H_{z_0}(\infty) = 0$.

Change back: $R_1(\frac{1}{z - z_0}) = R(z)$, so

$R(z) - G_{z_0}(\frac{1}{z - z_0})$ has a zero at z_0 (not a pole).

$G_{z_0}(\frac{1}{z - z_0})$ - polynomial of $\frac{1}{z - z_0}$, only has a pole at z_0!

[illegible] order of z_0 as a pole

deg G_{z_0} — order of z_0 as a pole.

Now let $z_1, \dots, z_n$ be all the poles of R. Then $R(z) - G_\infty(z) - \sum_{k=1}^{n} G_{z_k}\left(\frac{1}{z-z_k}\right)$ has no poles. So it is a constant, equal to zero at ∞. So

$R(z) = G_\infty(z) + \sum_{k=1}^{n} G_{z_k}\left(\frac{1}{z-z_k}\right)$ — partial fraction decomposition.